

REPRESENTATION OF STATE PROPERTY SYSTEMS

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ABSTRACT. A ‘state property system’ is the mathematical structure which models an arbitrary physical system by means of its set of states, its set of properties, and a relation of ‘actuality of a certain property for a certain state’. We work out a new axiomatization for standard quantum mechanics, starting with the basic notion of state property system, and making use of a generalization of the standard quantum mechanical notion of ‘superposition’ for state property systems.

1. INTRODUCTION

In standard quantum mechanics a state $p_{\bar{c}}$ of a quantum entity S is represented by the one dimensional subspace or the ray \bar{c} of a separable complex Hilbert space \mathcal{H} . An experiment e_A testing an observable A is represented by a self adjoint operator \mathcal{A} on \mathcal{H} , and the set of outcomes of this experiment e_A is the spectrum $\text{spec}(\mathcal{A})$ of this self-adjoint operator \mathcal{A} . Measurable subsets $B \subset \text{spec}(\mathcal{A})$ represent the events (in the sense of probability theory) of outcomes. The interaction of the experiment e_A with the physical entity being in state $p_{\bar{c}}$ is described in the following way: (1) the probability for a specific event $B \subset \text{spec}(\mathcal{A})$ to occur if the entity is in a specific state $p_{\bar{c}}$ is given by $\langle c, P_B(c) \rangle$, where P_B is the spectral projection corresponding to B , c is the unit vector in the ray \bar{c} representing state $p_{\bar{c}}$, and $\langle \cdot, \cdot \rangle$ is the inproduct in the Hilbert space \mathcal{H} ; (2) if the outcome is contained in B , the state $p_{\bar{c}}$ is changed to $p_{\bar{d}}$ where \bar{d} is the ray generated by $P_B(c)$.

Hence in standard quantum mechanics the states and experiments are represented by means of mathematical entities of a complex Hilbert space. The crucial role that is played by this complex Hilbert space is very much *ad hoc*, in the sense that there are no physically plausible reasons why the Hilbert space structure should be at the origin of both the structure of the state space, as well as the structure of the experiments.

This initiated the search for an axiomatic theory for quantum mechanics where the Hilbert space structure would be derived from more general and physically more plausible axioms. The area of forming physical models in the field of quantum mechanics is very large, and often involves philosophical problems of physics. Let us mention some of the most well known axiomatic approaches: the algebraic approach [1, 2, 3], where the basic notions are observables, the convexity approach [4, 5, 6, 7, 8], where the basic notion is the convex set of states, the empirical logic approach [9, 10, 11, 12, 13] where the authors start with primitive notions of an operation or a test, and the quantum logic approach [14, 15, 16, 17, 18, 19, 20] which starts with the set of experimental propositions.

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Due to the original focus [14] on the collection of ‘experimental propositions’ of a physical entity—with the conviction that such an ‘experimental proposition’ would be a good basic concept—most of the later axiomatics were constructed taking as their basic concept the set \mathcal{L} of experimental propositions concerning an entity S . The first breakthrough came with a theorem of Constantin Piron, who proved that if \mathcal{L} is a complete [axiom 1], orthocomplemented [axiom 2] atomic [axiom 3] lattice, which is weakly modular [axiom 4] and satisfies the covering law [axiom 5], then each irreducible component of the lattice \mathcal{L} can be represented as the lattice of all ‘biorthogonal’ subspaces of a vector space V over a division ring K (with some other properties satisfied that we shall not explicit here) [17]. Such a vector space is called an ‘orthomodular space’ and also sometimes a ‘generalized Hilbert space’. It can be proven that an infinite dimensional orthomodular space over a division ring which is the real or complex numbers, or the quaternions, is a Hilbert space. For a long time there did not even exist any other example of an infinite dimensional orthomodular space. The search for a further characterization of the real, complex or quaternionic Hilbert space started [21]. Then Hans Keller constructed a non classical orthomodular space [22], and recently Maria Pia Solèr proved that any orthomodular space that contains an infinite orthonormal sequence is a real, complex or quaternionic Hilbert space [23, 24]. It is under investigation in which way this result of Solèr can be used to formulate new physically plausible axioms [24, 25, 26, 27].

The axiomatic approach, apart from delivering an axiomatic foundations for quantum mechanics, has been used fruitfully to study concrete problems in quantum mechanics. As an example we mention the problem of the description of joint quantum entities, and the problems of entanglement, non-locality and appearance of the complex numbers in quantum mechanics. Most recently this problem has been studied within the axiomatic approach with very interesting results [28, 29, 30, 31, 32, 33, 34].

Next to the idea of finding axioms that introduce the Hilbert space structure step by step, was the attempt of founding the basic notions for this axiomatics in a physically clear and operational way. ‘Operationality’ means that the axioms should be introduced in such a way that they can be related to ‘real physical operations’ that can be performed in the laboratory.

The approaches that have tried to formulate quantum mechanics operationally are, the Geneva-Brussels approach [17, 18, 35, 36, 37, 38, 39, 40, 41], the Amherst approach [9, 10, 11, 12, 13], and the Marburg approach [42, 43]. In the present article we elaborate further on the Geneva-Brussels approach. Already in the last versions of the formalism that were presented in this approach the power of making a good distinction between the mathematical aspects of the formalism and its physical foundations had been identified [44, 45]. Let us explain more concretely what we mean. In the older founding papers of the Geneva-Brussels approach [35, 36, 37, 38, 39, 40, 41], although the physical foundation of the formalism is defined in a clear way, and the resulting mathematical structures are treated rigorously, it is not always clear what are the ‘purely mathematical’ properties of the structures that are at the origin of the results. That is the reason that in more recent work on the formalism we have made an attempt to divide up the physical foundation and the resulting mathematical structure as much as possible. We first explain in which way certain aspects of the mathematical structure arise from the

physical foundation, but then, in a second step, define these aspects in a strictly mathematical way, such that propositions and theorems can be proven, ‘only’ using the mathematical structure without physical interpretation. Afterwards, the results of these propositions and theorems can then be interpreted in a physical way again. This not only opens the way for mathematicians to start working on the structures, but also lends a greater axiomatic strength to the whole approach on the fundamental level. More concretely, it is the mathematical structure of a ‘state property system’ that has been identified to be the proper mathematical structure of the Geneva-Brussels approach, *i.e.* the structure used to describe a physical entity by means of its states and properties [44, 45, 46]. This step turned out to be fruitful from the start, since we could prove that a state property system as a mathematical structure is isomorphic to a closure space [44, 45, 46]. This means that the mathematics of closure spaces can be translated to the mathematics of state property systems, and in this sense becomes relevant for the foundations of quantum mechanics. The step of dividing up the mathematics from the physics in a systematic way also led to a scheme to derive the morphisms for the structures that we consider from a covariance principle rooted in the relation of a subentity to the entity of which it is a subentity [45, 46]. This paved the way to a categorical study of the mathematical structures involved.

Not only was it possible to connect with a state property system a closure space in an isomorphic way, but, after we had introduced the morphisms starting from a merological covariance principle, it was possible to prove that the category of state property systems and their morphisms, that we have named **SP**, is equivalent with the category of closure spaces and continuous functions, denoted by **Cls** [45, 46]. More specifically we could prove that **SP** is the amnesic modification of **Cls** [47].

It could be proven that some of the axioms of axiomatic quantum mechanics [17, 36, 37] correspond to separation properties of the corresponding closure spaces [48]. More concretely, the axiom of state determination in a state property system [44] is equivalent with the T_0 separation axiom of the corresponding closure space [48, 49], and the axiom of atomicity in a state property system [44] is equivalent with the T_1 separation axiom of the corresponding closure space [50, 51]. More recently it has been shown that ‘classical properties’ [36, 38, 40, 41] of the state property system correspond to clopen (open and closed) sets of the closure space [52, 53, 57], and, explicitly making use of the categorical equivalence, a decomposition theorem for a state property system into its nonclassical components can be proved that corresponds to the decomposition of the corresponding closure space into its connected components [52, 53, 57].

In the present article we put forward a new axiomatization for standard quantum mechanics, starting with the basic notion of ‘state property system’, and founded on the concept of ‘superposition’, that started in the quantum logic approach and was developed in [55]. The general reason for introducing the new axiomatization is to put under one roof the Geneva-Brussels approach in its recent form and the quantum logic approach and to combine the algebraic approach and the probabilistic approach. We also wish to find out how the recent development in projective geometry (see [56]) can be reflected in the axiomatization. A more specific reason for this new axiomatization is to take it as a mathematical basis for further research into the problem of the description of joint quantum systems. Both authors have done extensive research on the problem of the description of joint quantum

systems [57, 58, 59, 60, 37, 61, 62, 63, 64, 65]. One of the problems is that all type of product constructions on the level of the quantum logic structure give rise to a situation where the joint quantum entity only has product states of the subentities. On the level of the Hilbert space, the joint system of two quantum systems is described by means of the tensor product of the Hilbert spaces of the subsystems, and in this case there is an abundance of non-product states, giving rise to the well known phenomenon of quantum entanglement. We plan to study the still open problem of coupled physical systems with entanglement, by investigating in which way we can introduce ‘superpositions between product states’ by means of the notion of ‘superposition’ which we introduced in this axiomatization on the level of the quantum logic.

The notion of a ‘superposition of states’ was introduced by Varadarajan [66] for states as probability measures on quantum logics, i.e., orthomodular lattices. In the same sense it is also used in [55]. In the present paper, we use a more general frame of a state property system to introduce the notion of ‘superposition’. We use superpositions to create two kinds of closure operations. The first one, together with a few simple additional axioms, enables us to associate the structure of a projective geometry with our state property system. A very useful tool here is the material presented in the recent book by Faure and Frölicher [56]. The first of our closure operations based on superposition leads to the formation of subspaces of a projective geometry. The second of our closure operations enables us to characterize closed subspaces of the projective geometry. Probabilities enter into play in order to introduce orthocomplementation on a subset \mathcal{L}_0 of the lattice \mathcal{L} , and we show that \mathcal{L}_0 can be organized into a σ -orthomodular poset with an order determining set of probability measures, which are supported by elements of the set Σ . The set \mathcal{L}_0 may be interpreted as a set of measurable properties, and may depend on the present state of knowledge and experimental techniques. In the following parts of the article, conditions are found under which the orthocomplementation can be extended to the whole \mathcal{L} , and \mathcal{L} then becomes a complete, atomistic, orthocomplemented lattice. Moreover, \mathcal{L} can be related with the closed subspaces of the projective geometry via the so-called Cartan map. The notion of ‘superposition principle’ is introduced to obtain irreducibility of the projective geometry. More generally, sectors are introduced as the minimal subspaces in which the superposition principle holds, and their topological characterization as clopen subspaces is derived. In the following the classical properties (or the superselection rules) are specified, and it is shown that they correspond exactly to the central elements of the lattice \mathcal{L} . In the following, we study conditions under which our projective geometry may admit some deeper properties, described in [56], such as the Mackey property or to become an orthogeometry. Although not all of our axioms have a physical meaning, we try to specify simple axioms which enable us to obtain different stages of the projective geometry. Eventually we find conditions under which a vector space can be associated with our state property system, and we finish with an analogue of the famous Piron theorem.

2. STATE PROPERTY SYSTEMS AND SUPERPOSITION

Definition 1. [44, 45, 46] *We say that $(\Sigma, \mathcal{L}, \xi)$ is a state-property system if $(\Sigma, <)$ is a pre-ordered set, $(\mathcal{L}, <, \wedge, \vee)$ is a complete lattice with the greatest element I*

and the smallest element 0, and ξ is a function

$$(1) \quad \xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$$

such that for $p \in \Sigma$ and $(a_i)_i \subseteq \mathcal{L}$, we have

$$(2) \quad I \in \xi(p),$$

$$(3) \quad 0 \notin \xi(p),$$

$$(4) \quad a_i \in \xi(p) \ \forall i \Leftrightarrow \bigwedge_i a_i \in \xi(p) \text{ (for an arbitrary set of indices)}$$

and for $p, q \in \Sigma$ and $a, b \in \mathcal{L}$ we have

$$(5) \quad p < q \Leftrightarrow \xi(q) \subseteq \xi(p)$$

$$(6) \quad a \leq b \Leftrightarrow \forall r \in \Sigma : a \in \xi(r) \Rightarrow b \in \xi(r)$$

Elements of Σ are called *states*, elements of \mathcal{L} are called *properties*.

Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system. For $S \subseteq \Sigma$ define $S \mapsto \lambda(S)$ as follows. First define, for any $p, q \in \Sigma$,

$$(7) \quad \lambda\{p, q\} := \{s \in \Sigma : a \in \xi(p) \cap \xi(q) \Rightarrow a \in \xi(s)\}.$$

We will say that a subset $S \subseteq \Sigma$ is λ -closed if for any $p, q \in S$ we have $\lambda\{p, q\} \subseteq S$. Denote by $\mathcal{L}(\Sigma)$ the set of all λ -closed subsets. For any $P \subseteq \Sigma$, define

$$(8) \quad \lambda(P) := \bigcap \{G : G \in \mathcal{L}(\Sigma), P \subseteq G\}.$$

That is, $\lambda(P)$ is the intersection of all λ -closed subsets of Σ that contain P .

Lemma 1. (i) For every subset $P \subseteq \Sigma$, $\lambda(P) \in \mathcal{L}(\Sigma)$. (ii) A subset $S \subseteq \Sigma$ is λ -closed if and only if $S = \lambda(S)$.

Proof. (i) Let $p, q \in \lambda(P)$, then $p, q \in G$ for every $P \subseteq G \in \mathcal{L}(\Sigma)$. Therefore $\lambda\{p, q\} \subseteq G$ for every such G , and consequently $\lambda\{p, q\} \subseteq \lambda(P)$.

(ii) If $S = \lambda(S)$, then $S \in \mathcal{L}(\Sigma)$ by (i). If S is λ -closed, then clearly, S is the smallest λ -closed subset of Σ containing S , hence $\lambda(S) = S$. \square

That is,

$$(9) \quad \mathcal{L}(\Sigma) = \{S \subseteq \Sigma : S = \lambda(S)\}.$$

Proof of the following statement is immediate.

Lemma 2. The mapping $\lambda : P \mapsto \lambda(P)$ satisfies the following properties:

$$(C1) \quad P \subseteq \lambda(P),$$

$$(C2) \quad P_1 \subseteq \lambda(P_2) \Rightarrow \lambda(P_1) \subseteq \lambda(P_2).$$

We recall that a map $C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfying conditions (C1) and (C2) is a *closure operator* on the set X ([56, Def. 3.1.1]). Consequently, λ is a closure operator on the set Σ .

According to [56, Remark 3.1.2], the following conditions are satisfied.

$$(1^0) \quad \lambda(A \cup B) = \lambda(\lambda(A) \cup B) = \lambda(\lambda(A) \cup \lambda(B)),$$

$$(2^0) \quad \lambda(A \cap B) \subseteq \lambda(\lambda(A) \cap B) \subseteq \lambda(A) \cap \lambda(B).$$

More generally,

$$(3^0) \quad \lambda(\bigcup \mathcal{A}) = \lambda(\bigcup \lambda(\mathcal{A})),$$

$$(4^0) \quad \lambda(\bigcap \mathcal{A}) \subseteq \bigcap \lambda(\mathcal{A}),$$

where $\mathcal{A} \subseteq \mathcal{P}(\Sigma)$ is an arbitrary subset and $\lambda(\mathcal{A})$ denotes the set $\{\lambda(A) : A \in \mathcal{A}\}$.

Recall that a system \mathcal{S} of sets is an *intersection system* if $\mathcal{A} \subseteq \mathcal{S}$ implies $\bigcap \mathcal{A} \in \mathcal{S}$. By [56, Prop. 3.1.4], $\mathcal{L}(\Sigma)$ is an intersection system.

Definition 2. We say that a state-property system $(\Sigma, \mathcal{L}, \xi)$ satisfies property

- (A) if there are at least two distinct states $r, s \in \Sigma$ and for all $p, q \in \Sigma$, $\xi(p) \subset \xi(q) \Rightarrow p = q$.

Property (A) implies that $\xi : \Sigma \rightarrow \mathcal{P}(\mathcal{L})$ is injective. By [46, Proposition 14], the pre-order $<$ on Σ defined by (5) of Definition 1 is a partial order. Property (A) implies that Σ has only a trivial order $p < q$ iff $p = q$.

Recall that a closure operator C on a set X is called *simple* if it satisfies the additional axiom:

- (C5) $C(\emptyset) = \emptyset$ and $C(x) = \{x\}$ for every $x \in X$.

(We write $C(x)$ instead of $C(\{x\})$). If X contains at least two different points, then the second property in (C5) implies the first one. Indeed, $\emptyset \subset \{x\}, \emptyset \subset \{y\}$ implies $C(\emptyset) \subset C(x) \cap C(y) = \{x\} \cap \{y\} = \emptyset$.

Lemma 3. Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system such that there are at least two different states $p, q \in \Sigma$. Then the closure operator λ is simple if and only if property (A) of Definition 2 holds.

Proof. If (A) holds, then for every $p \in \Sigma$,

$$\begin{aligned} \lambda\{p\} &= \{s \in \Sigma : a \in \xi(p) \Rightarrow a \in \xi(s)\} \\ &= \{s \in \Sigma : \xi(p) \subset \xi(s)\} = \{p\}. \end{aligned}$$

If $\lambda(\emptyset)$ contains r , then $\lambda(\emptyset) \subseteq \lambda(q) = \{q\}$ implies $r \in \{q\}$, hence $r = q$ for all q , a contradiction.

Conversely, if λ is simple and $\xi(p) \subseteq \xi(q)$ for $p \neq q$, then

$$\lambda(p) = \{s \in \Sigma : \xi(p) \subseteq \xi(s)\} \ni q$$

contradicting $\lambda(p) = \{p\}$. □

Proposition 1. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system satisfying (A). Then \mathcal{L} is a complete atomistic lattice with the lattice operations

$$\bigwedge S_\alpha = \bigcap S_\alpha, \bigvee S_\alpha = \lambda\left(\bigcup S_\alpha\right).$$

Proof. Follows by [56, prop. 3.1.4]. □

In what follows, we introduce the notion of a superposition of states in analogy with [19].

Definition 3. A state $p \in \Sigma$ is a *superposition* of a set of states S , $S \subseteq \Sigma$, if for each $a \in \mathcal{L}$, $a \in \xi(s)$ for all $s \in S$ implies $a \in \xi(p)$, i.e. if $\bigcap_{s \in S} \xi(s) \subseteq \xi(p)$.

For $S \subseteq \Sigma$, define

$$(10) \quad \bar{S} = \{p \in \Sigma : \bigcap_{s \in S} \xi(s) \subseteq \xi(p)\}.$$

That is, \bar{S} is the set of all superpositions of states in S . Obviously, for arbitrary $p, q \in \Sigma$,

$$(11) \quad \{p, q\}^- = \lambda\{p, q\}.$$

Definition 4. A state $p \in \Sigma$ is a minimal superposition of a subset $S \subseteq \Sigma$ if

- (i) $p \in \bar{S}$,
- (ii) $p \notin \bar{Q}$ for any proper subset $Q \subseteq S$.

For example, if (A) holds, then $s \in \{p, q\}^-$ is a minimal superposition iff $s \neq p$, $s \neq q$.

Definition 5. Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system.

(1) We will say that a minimal superposition principle (MSP) holds for $(\Sigma, \mathcal{L}, \xi)$ if for every subset $S \subseteq \Sigma$ and for every minimal superposition p of S ,

$$(12) \quad \{S_1 \cup p\}^- \cap \bar{S}_2 \neq \emptyset$$

whenever S_1, S_2 are proper subsets of S such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$.

(2) We will say that a finite MSP (f-MSP) holds for $(\Sigma, \mathcal{L}, \xi)$ if (12) holds for every finite subset $S \subseteq \Sigma$.

(3) We will say that an n -MSP holds for $(\Sigma, \mathcal{L}, \xi)$ if (12) holds for every subset $S \subseteq \Sigma$ with the cardinality at most n .

For example, the 3-MSP holds for a state-property system $(\Sigma, \mathcal{L}, \xi)$ iff for every $p, q, r, s \in \Sigma$ (not necessarily all different), if $p \in \{q, r, s\}^-$ is a minimal superposition, then $\{p, q\}^- \cap \{r, s\}^- \neq \emptyset$ and $\{p, q, r\}^- \cap \{s\}^- \neq \emptyset$. Clearly, MSP implies f-MSP, which in turn implies n-MSP for every $n \in \mathbb{N}$. Observe also that if (A) holds, then 2-MSP is the following exchange property : if $r \in \{p, q\}^-$ and $r \neq p, q$, then $p \in \{r, q\}^-$.

Theorem 1. Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system with properties (A) and 3-MSP. Then the operator $*$: $\Sigma \times \Sigma \rightarrow \mathcal{P}(\Sigma)$ defined by $p * q = \lambda\{p, q\}$ has the following properties:

- (P1) $p * p = \{p\}$,
- (P2) $p \in p * q$ for all $p, q \in \Sigma$,
- (P3) $p \in q * r$ and $r \in s * t$ and $p \neq s$ imply $(p * s) \cap (q * t) \neq \emptyset$

That is, the system $(\Sigma, *)$ is a projective geometry [56].

Proof. (P1) By (A), $\lambda\{p, p\} = \{p\}$.

(P2) Clearly, $p * q = \lambda\{p, q\} \supseteq \{p\}$.

(P3) From $p \in \{q, r\}^-$ and $r \in \{s, t\}^-$ we obtain $p \in \{q, s, t\}^-$. If $p \in q * s$ resp. $p \in s * t$, then either $p = q$, resp. $p = t$, or 2-MSP implies that $q \in p * s$, resp. $t \in p * s$. In every case, (P3) is satisfied. It remain the case that either $p \in q * t$ or p is a minimal superposition of $\{q, s, t\}$. In the first case, $p \in (p * s) \cap (q * t)$ holds by (P2). In the second case the statement follows by 3-MSP. \square

By [56, Def. 2.3.1], the λ -closed subsets of Σ coincide, under the suppositions of Theorem 1, with the subspaces of the projective geometry $(\Sigma, *)$ associated with $(\Sigma, \mathcal{L}, \xi)$. Consequently, we have the following.

Theorem 2. Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system satisfying conditions (A) and 3-MSP. Then $\mathcal{L}(\Sigma)$ is a projective lattice, i.e., a complete atomistic meet-continuous modular lattice.

Therefore we will call the elements of $\mathcal{L}(\Sigma)$ the *subspaces* of Σ . From the next theorem we can derive what properties satisfies the closure operator λ on the system $(\Sigma, *)$ with $p * q = \lambda\{p, q\}$. (We write $\lambda(x)$ instead of $\lambda(\{x\})$ and $\lambda(A \cup x)$ instead of $\lambda(A \cup \{x\})$.)

Theorem 3. *Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system such that $(\Sigma, *)$ with $p * q = \lambda\{p, q\}$ is a projective geometry, i.e., properties (P1), (P2) and (P3) are satisfied. Then the closure operator λ satisfies the following conditions.*

- (C3) $x \in \lambda(A)$ implies $x \in \lambda(B)$ for some finite subset $B \subseteq A$, i.e., λ is finitary.
- (C4) $x \notin \lambda(A)$ and $x \in \lambda(A \cup y)$ imply $y \in \lambda(A \cup x)$, i.e., λ satisfies the exchange property.
- (C5) $\lambda(\emptyset) = \emptyset$ and $\lambda(x) = \{x\}$, i.e., λ is simple.
- (C6) $\lambda(A \cup B) = \bigcup \{\lambda\{x, y\} : x \in \lambda(A) \text{ and } y \in \lambda(B)\}$ for every $A, B \subseteq \Sigma$.

Proof. See [56, Th. 3.3.4]. \square

Notice that (C6) is called a *projective law*. According to [56, Lemma 3.3.2], if a closure operator $C : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfies the projective law, then for any nonempty subset $A \subseteq X$ and any $b \in X$ one has

$$(C7) \quad C(A \cup b) = \bigcup \{C(x, b) : x \in C(A)\}.$$

Moreover, the converse holds provided the closure operator C satisfies (C3), i.e. is finitary. The following proposition follows by [56, Proposition 3.3.4].

Proposition 2. *Let the closure operator $\lambda : \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$ satisfy properties (C4), (C5) and (C7). The couple $(\Sigma, *)$ where $p * q = \lambda\{p, q\}$ is a projective geometry.*

Now we will study connections between the mappings $A \mapsto \lambda(A)$ and $A \mapsto \bar{A}$, $A \subseteq \Sigma$. First we prove the following properties of $A \mapsto \bar{A}$.

Lemma 4. *Let $(\Sigma, \mathcal{A}, \xi)$ be a state-property system. The operator $A \mapsto \bar{A}$ satisfies the following properties for every $A, B \subseteq \Sigma$.*

- (i) $A \subseteq \bar{A}$.
- (ii) $A \subseteq \bar{B} \Rightarrow \bar{A} \subseteq \bar{B}$.
- (iii) $\bar{A} \in \mathcal{L}(\Sigma)$.

Proof. (i) Follows directly from the definition.

(ii) Let $p \in \bar{A}$, i.e., $\bigcap_{s \in A} \xi(s) \subseteq \xi(p)$. $A \subseteq \bar{B}$ implies that for every $s \in A$, $\bigcap_{q \in B} \xi(q) \subseteq \xi(s)$, so that $\bigcap_{q \in B} \xi(q) \subseteq \bigcap_{s \in A} \xi(s) \subseteq \xi(p)$. Hence $p \in \bar{B}$, and so $\bar{A} \subseteq \bar{B}$.

(iii) From $\lambda\{p, q\} = \{p, q\}^- \subseteq \bar{A}$ for every $p, q \in A$ we see that \bar{A} is λ -closed, i.e., $\bar{A} = \lambda(\bar{A})$. \square

Observe that (i) and (ii) in Lemma 4 coincide with the properties (C1) and (C2), respectively, so that $A \mapsto \bar{A}$ is a closure operator. Let us denote by $\mathcal{F}(\Sigma)$ the set of superposition-closed subsets of Σ , that is,

$$(13) \quad \mathcal{F}(\Sigma) := \{S \subset \Sigma : \bar{S} = S\}.$$

Proposition 3. *Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system satisfying condition (A). Then the set $\mathcal{F}(\Sigma)$ is a complete atomistic lattice. Moreover, if $S_i \in \mathcal{F}(\Sigma)$, $i \in I$, for any index set I , then $\bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i$, and $\bigvee_{i \in I} S_i = (\bigcup_{i \in I} S_i)^-$.*

Proof. If $S \subset \Sigma$, $S = \emptyset$ or $S = \{s\}$, then $S = \bar{S}$ by condition (A). So one-element sets belong to $\mathcal{F}(\Sigma)$ which are atoms in $\mathcal{F}(\Sigma)$. From the properties of closure operators ([56, Prop. 3.1.4]), we get $\bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i$, and $\bigvee_{i \in I} S_i = (\bigcup_{i \in I} S_i)^-$. \square

Theorem 4. *Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system such that condition (A) is satisfied.*

(i) If 3-MSP holds, then for every $p, q, s \in \Sigma$,

$$(14) \quad \{p, q, s\}^- = \lambda\{p, q, s\}$$

(ii) If f-MSP holds, then for every finite subset $A = \{s_1, s_2, \dots, s_n\} \subseteq \Sigma$,

$$(15) \quad \lambda(A) = \bar{A}.$$

Proof. (i) For every $p, q \in \Sigma$, $\lambda\{p, q\} = \{p, q\}^-$, and by Lemma 4 (iii), $\lambda\{p, q, s\} \subseteq \{p, q, s\}^-$ for every $p, q, s \in \Sigma$. To prove the converse inclusion, let $t \in \{p, q, s\}^-$. If $t \in \{p, q\}^-$, then $t \in \lambda\{p, q\} \subseteq \lambda\{p, q, s\}$. Hence we may assume that t is a minimal superposition. Then by 3-MSP, there is $r \in \{p, t\}^- \cap \{q, s\}^-$. By 2-MSP, $t \in \lambda\{r, p\} \subseteq \lambda\{p, q, s\}$. This implies that $\{p, q, s\}^- \subseteq \lambda\{p, q, s\}$.

(ii) We will proceed by induction. For $n=2$, the statement holds. Assume that the statement holds for every $k \leq n$, $k, n \in \mathbb{N}$. Let $A = \{s_1, s_2, \dots, s_n, s_{n+1}\}$, and assume that $t \in \bar{A}$ is a minimal superposition. By f-MSP and induction hypothesis, there is $r \in \{t, s_{n+1}\}^- \cap \{s_1, \dots, s_n\}^- \subseteq \lambda\{s_1, \dots, s_n\}$. Now $t \in \lambda\{r, s_{n+1}\} \subseteq \lambda((\lambda\{s_1, \dots, s_n\} \cup s_{n+1}) \subseteq \lambda(A)$. If t is not a minimal superposition, there is a subset $B \subseteq A$ such that $t \in \bar{B} = \lambda(B) \subseteq \lambda(A)$ by induction hypothesis. Hence $\bar{A} \subseteq \lambda(A)$. The converse holds by Lemma 4 (iii). \square

3. PROBABILITY MEASURES AND ORTHOCOMPLEMENTATION

Let $(\Sigma, \mathcal{L}, \xi)$ be a state-property system. Let there be a subset $\mathcal{L}_0 \subset \mathcal{L}$ such that \mathcal{L}_0 contains 0 and 1, and let there be a mapping $\mu : \Sigma \times \mathcal{L}_0 \rightarrow [0, 1]$, $(p, a) \mapsto \mu_p(a)$, where $[0, 1]$ is the unit interval of the reals, such that

- (Oi) $\mu_p(a) = 1$ iff $a \in \xi(p)$ ($a \in \mathcal{L}_0$),
- (Oii) $a \leq b$ implies $\mu_p(a) \leq \mu_p(b)$ ($a, b \in \mathcal{L}_0$),
- (Oiii) If $(a_i)_{i=1}^\infty \subset \mathcal{L}_0$ is a sequence such that for all i, j , and every $p \in \Sigma$,

$$\mu_p(a_i) + \mu_p(a_j) \leq 1,$$

then there is $b \in \mathcal{L}_0$ such that

$$\mu_p(b) + \sum_{i=1}^\infty \mu_p(a_i) = 1.$$

Clearly, $\mu_p(I) = 1$ and $\mu_p(0) = 0$ for all $p \in \Sigma$. Define a relation $\perp \subset \mathcal{L}_0 \times \mathcal{L}_0$ by setting $a \perp b$ iff $\mu_p(a) + \mu_p(b) \leq 1$ for all $p \in \Sigma$. We will say that a and b are *orthogonal* if $a \perp b$.

Lemma 5. *Let $(\mathcal{L}, \Sigma, \xi)$ be a state-property system. Let $\mathcal{L}_0 \subset \mathcal{L}$ and $\mu : \Sigma \times \mathcal{L}_0 \rightarrow [0, 1]$ satisfy the assumptions (Oi) - (Oiii). Then*

- (i) $\mu_p(a) \leq \mu_p(b)$ for every $p \in \Sigma$ implies $a \leq b$.
- (ii) $\mu_p(a) = \mu_p(b)$ for all $p \in \Sigma$ if and only if $a = b$.
- (iii) For every $a \in \mathcal{L}_0$ there is a unique element $a' \in \mathcal{L}_0$ such that $\mu_p(a) + \mu_p(a') = 1$ for all $p \in \Sigma$. Moreover, the mapping $a \mapsto a'$ is an orthocomplementation in \mathcal{L}_0 , i.e., (1) $a \leq b \Rightarrow b' \leq a'$, (2) $a'' := (a')' = a$, (3) $a \vee_0 a' = I$, $a \wedge_0 a' = 0$, where \vee_0 and \wedge_0 denote the supremum and infimum in \mathcal{L}_0 , respectively.
- (iv) For every sequence $(a_i)_{i=1}^\infty$ of mutually orthogonal elements in \mathcal{L}_0 , their supremum $a = \bigvee_{i=1}^\infty a_i$ exists in \mathcal{L}_0 , and coincides with the supremum of $(a_i)_{i=1}^\infty$ in \mathcal{L} .

Proof. (i) If $\mu_p(a) \leq \mu_p(b)$ for every $p \in \Sigma$, then $\mu_p(a) = 1 \Rightarrow \mu_p(b) = 1$, hence by (Oi), $a \in \xi(p) \Rightarrow b \in \xi(p)$, which implies $a \leq b$.

(ii) follows by (Oii) and (i).

(iii) Let $a \in \mathcal{L}_0$, and consider the sequence $(a_i)_{i=1}^\infty$ where $a_1 = a$, $a_i = 0$, $i = 2, 3, \dots$. By (Oiii), there is $b \in \mathcal{L}_0$ such that $\mu_p(b) + \mu_p(a) = 1$ for all $p \in \Sigma$, i.e., $\mu_p(b) = 1 - \mu_p(a)$ for all $p \in \Sigma$. Hence we may put $a' = b$. By (ii), a' is uniquely defined. Now we prove that $a \mapsto a'$ is an orthocomplementation.

(1) $a \leq b \Rightarrow \mu_p(a) \leq \mu_p(b)$ for all $p \in \Sigma$, which implies $\mu_p(b') = 1 - \mu_p(b) \leq 1 - \mu_p(a) = \mu_p(a')$ for all $p \in \Sigma$, which by (ii) entails $b' \leq a'$.

(2) $\mu_p((a')') = 1 - \mu_p(a') = 1 - (1 - \mu_p(a)) = \mu_p(a)$ for all $p \in \Sigma$, which entails $a'' = a$.

(3) Let $c \in \mathcal{L}_0$ be such that $a \geq c, a' \geq c$. From $\mu_p(a) + \mu_p(a') = 1$ and $\mu_p(a) = 1$ iff $a \in \xi(p)$, it follows that $a \in \xi(p) \Rightarrow a' \notin \xi(p)$, and vice versa. Hence $a \in \xi(p)$ and $a' \in \xi(p)$ happens for no $p \in \Sigma$, which entails, by Definition 1, that $a \wedge a' = 0$ in \mathcal{L} . Since $0 \in \mathcal{L}_0$, the infimum of a and a' in \mathcal{L}_0 is 0.

Properties (1) and (2) imply de Morgan laws in \mathcal{L}_0 : $a \vee_0 b$ exists, then $(a \vee_0 b)' = a' \wedge_0 b'$, and $(a \wedge_0 b)' = a' \vee_0 b'$ in the sense that if one side exists, so does the other, and they are equal. Therefore for every $a \in \mathcal{L}_0$, $a' \wedge_0 a'' = 0$ implies $(a' \wedge_0 a'')' = a \vee_0 a' = I$.

(iv) Let $(a_i)_{i=1}^\infty$ be a sequence of pairwise orthogonal elements of \mathcal{L}_0 . Let b be the element from (Oiii). Put $a := b'$, then for every $p \in \Sigma$ we have $\mu_p(a) = \sum_{i=1}^\infty \mu_p(a_i)$. It follows that $\mu_p(a_i) \leq \mu_p(a)$ for all $i \in \mathbb{N}$, and for all $p \in \Sigma$. Hence a is an upper bound of $a_i, i = 1, 2, \dots$. Let $c \in \mathcal{L}_0$ be any other upper bound of $a_i, i = 1, 2, \dots$. Then $a_i \leq c$ for all i implies that c', a_1, a_2, \dots are mutually orthogonal. By (Oiii), there is an element $d \in \mathcal{L}_0$ such that for every $p \in \Sigma$,

$$\mu_p(d) = \mu_p(c') + \sum_{i=1}^\infty \mu_p(a_i) = \mu_p(c') + \mu_p(a)$$

From this we obtain $\mu_p(c) = \mu_p(d') + \mu_p(a)$, which entails by (i) that $a \leq c$.

Let u be the supremum of $(a_i)_{i=1}^\infty$. We have $\forall p \in \Sigma, u \in \xi(p)$ if and only if $a_i \in \xi(p)$ for some i . But then $u \in \xi(p)$ if and only if $a \in \xi(p)$, which entails that $u = a$. \square

We will say that a set \mathcal{F} of functions $f : L \rightarrow [0, 1]$ defined on a partially ordered set L is *order determining* if $a \leq b \Leftrightarrow \forall f \in \mathcal{F}, f(a) \leq f(b)$.

Theorem 5. *Let $(\mathcal{L}, \Sigma, \xi)$ be a state-property system, $\mathcal{L}_0 \subseteq \mathcal{L}$, and let $\mathcal{M} := \{\mu : \Sigma \times \mathcal{L}_0 \rightarrow [0, 1]\}$ satisfy conditions (Oi)-(Oiii). Then the set \mathcal{L}_0 is a σ -orthomodular poset and the set \mathcal{M} is order determining for \mathcal{L}_0 . Moreover, for every $a \in \mathcal{L}_0, a \neq 0$, there is $p \in \Sigma$ such that $\mu_p(a) = 1$.*

Proof. By definition, the set \mathcal{L}_0 with the ordering inherited from \mathcal{L} is a partially ordered set. By Lemma 5, \mathcal{L}_0 is an orthocomplemented set such that the supremum of every pairwise orthogonal sequence exists in \mathcal{L}_0 . Moreover, \mathcal{M} is ordering for \mathcal{L}_0 . Assume $a \leq b, a, b \in \mathcal{L}_0$. Then $\forall p \in \Sigma, \mu_p(a) \leq \mu_p(b)$ implies $\mu_p(a) + \mu_p(b') \leq 1$, so that $a \vee b'$ exists in \mathcal{L}_0 and $\mu_p(a \vee b') = \mu_p(a) + \mu_p(b')$ for all $p \in \Sigma$, which entails that $\mu_p(b) = \mu_p(a) + \mu_p(a' \wedge b)$ for all $p \in \Sigma$, hence $\mu_p(b) = \mu_p(a \vee (a' \wedge b))$ for all $p \in \Sigma$, so by Lemma 5 (ii), $b = a \vee (a' \wedge b)$, which is the orthomodular law. Hence \mathcal{L}_0 is a σ -orthocomplete orthomodular poset.

Let $a \in \mathcal{L}$, $a \neq 0$, and assume that $\forall p \in \Sigma$, $a \notin \xi(p)$. Then the implication

$$\forall r \in \Sigma : a \in \xi(r) \Rightarrow 0 \in \xi(r)$$

holds, which by (5) of Definition 1 means that $a = 0$, a contradiction. If $0 \neq a \in \mathcal{L}_0$, then $a \in \xi(p)$ for at least one $p \in \Sigma$ means that $\mu_p(a) = 1$. \square

From now on, we will write $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ to denote a state property system for which there is $\mathcal{L}_0 \subset \mathcal{L}$ with a system of functions $\mu_s, s \in \Sigma$ such that conditions (Oi), (Oii) and (Oiii) are satisfied.

Definition 6. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be given. We will say that μ_p has a support (in \mathcal{L}_0) if there is an element $b \in \mathcal{L}$ ($b \in \mathcal{L}_0$) such that $\forall a \in \mathcal{L}_0$, $\mu_p(a) = 1$ iff $b \leq a$.

Clearly, if a support exists, it is unique.

Proposition 4. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system, satisfying condition (A). For $p \in \Sigma$, let $a_p := \bigwedge \{a : a \in \xi(p)\}$. Then $a_p, p \in \Sigma$, coincide with the atoms in \mathcal{L} . Moreover, $a \in \xi(p)$ if and only if $a_p \leq a$.

Proof. Observe that condition (A) implies also condition

$$(A') \text{ for all } p \in \Sigma, \text{ the element } a_p = \bigwedge \{a : a \in \xi(p)\} \neq 0.$$

Indeed, by Definition 1, $\bigwedge \{a : a \in \xi(p)\} \in \xi(p)$, and $0 \notin \xi(p)$. Hence $a_p \in \xi(p)$, and clearly, a_p is the smallest element in $\xi(p)$. Assume that $a_p \in \xi(r)$, $r \in \Sigma$. Now $a_p \leq a$ for all $a \in \xi(p)$ implies that $a \in \xi(r)$ for all $a \in \xi(p)$, hence $\xi(p) \subset \xi(r)$. By condition (A) then $p = r$.

Assume $b \leq a_p$, $b \neq 0$, then $\exists r, b \in \xi(r)$ and we have

$$\begin{aligned} \forall r \in \Sigma, b \in \xi(r) &\Rightarrow a_p \in \xi(r) \Rightarrow a \in \xi(r) \forall a \in \xi(p) \\ \xi(p) \subset \xi(r) &\Rightarrow p = r \Rightarrow b \in \xi(p) \Rightarrow a_p \leq b. \end{aligned}$$

This proves that a_p is an atom in \mathcal{L} .

Now let a be an atom of \mathcal{L} . Then there is $r \in \Sigma$ with $a \in \xi(r)$, hence $a_r \leq a$. Since a_r is an atom, $a_r = a$. \square

Notice that under conditions of Proposition 4, the element a_p is a support of μ_p .

Theorem 6. Under the suppositions of Proposition 4, \mathcal{L} is an atomistic lattice.

Proof. Let $b \in \mathcal{L}$, put $c = \bigvee \{a_s : b \in \xi(s)\}$. Then clearly $c \leq b$, and if $b \in \xi(p)$, then $a_p \leq c$ implies $c \in \xi(p)$, therefore $b = c$. \square

Theorem 7. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be a state property system satisfying condition (A) and

- (B) For every $s \in \Sigma$, a_s belongs to \mathcal{L}_0 .
- (C) For every $b \in \mathcal{L}$, $b = \bigwedge \{a'_s : b \leq a'_s\}$.

Then \mathcal{L} with the mapping $b' := \bigvee \{a_s : b \leq a'_s\}$ is a complete, atomistic, orthocomplemented lattice.

Proof. Owing to Theorem 6, it suffices to prove that \mathcal{L} is orthocomplemented. (i) If $b \leq c$, then $\{a_s : c \leq a'_s\} \subset \{a_s : b \leq a'_s\}$, which by (C) implies $c' \leq b'$. (ii) From $b' \leq a'_s$ iff $a_s \leq b$ we obtain that $(b')' = \bigvee \{a_s : b' \leq a'_s\} = \bigvee \{a_s : a_s \leq b\} = b$. It remains to prove that $b \wedge b' = 0$. Assume that $a_s \leq b, a_s \leq b'$. By (i) and (ii), $a_s \leq b \leq a'_s$, which contradicts (B), so $b \wedge b' = 0$. By duality we get $b \vee b' = I$. \square

Definition 7. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be given. We will say that p is orthogonal to q , $p, q \in \Sigma$, if there is $a \in \mathcal{L}_0$ such that $\mu_p(a) = 1$ and $\mu_q(a) = 0$ (equivalently, $p \perp q$ if $a \in \xi(p)$, $a' \in \xi(q)$). If p is orthogonal to q we will write $p \perp q$.

It is obvious that the relation \perp is symmetric and anti-reflexive. For $T \subset \Sigma$, we put $T' = \{p \in \Sigma : p \perp T\}$, where $p \perp T$ means that $p \perp t$ for all $t \in T$. Clearly, $\emptyset' = \Sigma$, $T \subseteq T''$ and $T_1 \subset T_2$ implies $T_1' \supset T_2' \forall T_1, T_2 \subset \Sigma$. If $s, p \in \Sigma$ have supports in \mathcal{L}_0 , then $s \perp p$ if and only if their supports are orthogonal.

Denote by \bar{T}^0 the set of all $s \in \Sigma$ such that $\forall a \in \mathcal{L}_0, a \in \xi(t) \forall t \in T \Rightarrow a \in \xi(s)$. That is, \bar{T}^0 is the set of all superpositions of $T \subset \Sigma$ with respect to \mathcal{L}_0 . Equivalently, $\bar{T}^0 = \{s \in \Sigma : \forall a \in \mathcal{L}_0, a_t \leq a \Rightarrow a_s \leq a\}$. Clearly, $\bar{T} \subset \bar{T}^0$.

Proposition 5. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be a state property system satisfying conditions (A),(B),(C). Then for every $T \subset \Sigma$, $\bar{T} = \bar{T}^0$.

Proof. It suffices to prove that $\bar{T}^0 \subset \bar{T}$. We have $s \in \bar{T}^0$ iff $\forall a \in \mathcal{L}_0, a_t \leq a \forall t \in T \Rightarrow a_s \leq a$. Let us take $b \in \mathcal{L}$, and assume that $a_t \leq b \forall t \in T$. By property (C), $b = \bigwedge \{a'_r : b \leq a'_r\}$, which yields $a_t \leq a'_r$ for all $t \in T$ and r such that $b \leq a'_r$. From $s \in \bar{T}^0$ we obtain that $a_s \leq a'_r$ for all corresponding r , and therefore $a_s \leq \bigwedge \{a'_r : b \leq a'_r\} = b$. In other words, $\bigvee_{t \in T} \xi(t) \subset \xi(s)$, hence $s \in \bar{T}$. \square

Proposition 6. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$. Suppose that (A), (B) are satisfied. Then for any $T \subset \Sigma$ we have $T'' = \bar{T}^0$.

Proof. We follow the proof of [55, Proposition 3.3.15]. We will identify μ_s with $s \in \Sigma$ and write $T(a) = k$ if $\mu_t(a) = k \forall t \in T$. First we show that $T' = \emptyset$ if and only if $\{a \in \mathcal{L}_0 : T(a) = 1\} = \{1\}$. Assume that $T' = \emptyset$ and let $a \in \mathcal{L}_0$ be such that $a \neq 1$ and $T(a) = 1$. Since $a' \neq 0$, there is $p \in \Sigma$ such that $p(a') = 1$. But then $p(a) = 0$, so that $a \in T'$, a contradiction. Now assume that $\{a \in \mathcal{L}_0 : T(a) = 1\} = \{1\}$ and also that $p \in T'$. Then for the supports we have $a_p \perp a_t \forall t \in T$. Hence $t(a'_p) = 1$ for all $t \in T$, which is again a contradiction.

To prove the equality $T'' = \bar{T}^0$, assume first that $T' = \emptyset$. We have already proved that then $\{a \in \mathcal{L}_0 : T(a) = 1\} = \{1\}$, which implies $\bar{T}^0 = \Sigma = T''$.

Assume that $T' \neq \emptyset$ and also that $p \in T'$. We will show that $p \in T''$. Assume that $q \in T'$, then $a_q \perp a_t \forall t \in T$, and hence $T(a'_q) = 1$. This implies $a'_q \in \bigcap_{t \in T} \xi(t)$, which implies that $a'_q \in \xi(p)$. This implies $q \perp p$, which implies that $\bar{T}^0 \subset T''$.

Assume that $p \in T''$ and also that $T(a) = 1$ for some $a \in \mathcal{L}_0$. Without loss of generality we may assume that $a \neq 1$. We have $a_q \leq a'$ iff $q(a') = 1$. But $q(a') = 1$ implies that $q \in T'$. This means that $q \perp p$, and so $a_p \perp a_q$ for all q such that $q(a') = 1$. Hence $a_q \leq a'$ implies $a_q \leq a'_p$, so that $a' \leq a'_p$, so that $p(a) = 1$. This shows that $p \in \bar{T}^0$ and this completes the proof. \square

As a corollary of Propositions 5 and 6, we obtain the following.

Corollary 1. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be a state property system satisfying (A),(B),(C). Then for every $T \subset \Sigma$, $\bar{T} = T''$.

Theorem 8. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be a state property system satisfying (A),(B). Define $\mathcal{F}^0(\Sigma) := \{S \subset \Sigma : S = \bar{S}^0\}$. Then the mapping $S \mapsto S'$ is an orthocomplementation on $\mathcal{F}^0(\Sigma)$. Consequently, $\mathcal{F}^0(\Sigma)$ is a complete, atomistic, orthocomplemented lattice. If also (C) holds, then $S \mapsto S'$ is an orthocomplementation on $\mathcal{F}(\Sigma)$, and $\mathcal{F}(\Sigma)$ is a complete, atomistic, orthocomplemented lattice.

Proof. It is easy to check that $S \mapsto \bar{S}^0$ is a closure operation, and hence $\mathcal{F}^0(\Sigma)$ is a complete lattice with lattice operations $S \wedge T = S \cap T$ and $S \vee T = (S \cup T)^{-0}$. Owing to property (A), $\mathcal{F}^0(\Sigma)$ is atomistic. To prove orthocomplementation, observe that $S \subset T \Rightarrow T' \subset S'$ and $S \wedge S' = \emptyset$ follow directly from the definition of the mapping $S \mapsto S'$. Property $S'' = S$ for $S \in \mathcal{F}^0(\Sigma)$ follows from Proposition 6. The remaining statement follows from Corollary 1. \square

Definition 8. Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property system. The map $\kappa : \mathcal{L} \rightarrow \mathcal{P}(\Sigma)$ defined by

$$(16) \quad \kappa(a) = \{p \in \Sigma : a \in \xi(p)\}$$

is called the Cartan map.

According to [46, Proposition 5], $\kappa : \mathcal{L} \rightarrow (\kappa(\mathcal{L}), \subset, \cap)$ has the following properties:

$$(17) \quad \kappa(1) = \Sigma,$$

$$(18) \quad \kappa(0) = \emptyset,$$

$$(19) \quad a \leq b \Leftrightarrow \kappa(a) \subset \kappa(b),$$

$$(20) \quad \kappa\left(\bigwedge_i a_i\right) = \bigcap_i \kappa(a_i).$$

That is, κ is an isomorphism of complete lattices. Moreover, by [46, Theorem 2], $\{\kappa(a) : a \in \mathcal{L}\}$ is an intersection system. Consequently, the operator $cl : Y \mapsto \bigcap\{\kappa(a) : Y \subset \kappa(a)\}$ is a closure operator [56].

Next lemma shows that $\kappa(a)$ is closed under superpositions.

Lemma 6. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. For all $a \in \mathcal{L}$, $\kappa(a) \in \mathcal{F}(\Sigma)$.

Proof. For every $a \in \mathcal{L}$ we have $\kappa(a) \subset \kappa(a)^-$. Observe that $p \in \kappa(a) \Leftrightarrow a \in \xi(p)$. Let $s \in \bar{\kappa}(a)$, then $\bigcap_{p \in \kappa(a)} \xi(p) \subset \xi(s)$ implies $a \in \xi(s)$, which means that $s \in \kappa(a)$. \square

Proposition 7. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system such that condition (A) is satisfied. Then $\kappa(\mathcal{L})$ and $\mathcal{F}(\Sigma)$ are isomorphic as complete atomistic lattices.

Proof. By Lemma 6, the range of κ is in $\mathcal{F}(\Sigma)$. By [46, Proposition 5], $\kappa(\mathcal{L})$ and $\mathcal{F}(\Sigma)$ are isomorphic as complete lattices. Let $a \in \mathcal{L}$ be an atom. By definition, $\kappa(a) = \{p \in \Sigma : a \in \xi(p)\}$. By (A), $a \in \xi(p)$ iff $a_p \leq a$, hence $a_p = a$ because a is an atom. By (A) we may conclude that $\kappa(a) = \{p\}$. \square

Theorem 9. Let $(\Sigma, \mathcal{L}, \mathcal{L}_0, \xi)$ be a state property system satisfying (A),(B). Then the mapping $\kappa : \mathcal{L}_0 \rightarrow \mathcal{F}(\Sigma)$, $a \mapsto \kappa(a)$ has the following properties:

- (i) If $a \wedge b$ exists in \mathcal{L}_0 , then $\kappa(a \wedge b) = \kappa(a) \wedge \kappa(b)$.
- (ii) For all $a \in \mathcal{L}_0$, $\kappa(a') = \kappa(a)'$.

Consequently, $\kappa(\mathcal{L}_0)$ and \mathcal{L}_0 are isomorphic as atomistic σ -orthomodular posets.

If also condition (C) is satisfied, then $\kappa(\mathcal{L})$ and $\mathcal{F}(\Sigma)$ are isomorphic as complete, atomistic orthocomplemented lattices.

Proof. (i) Suppose that $a \wedge b$ exists in \mathcal{L}_0 . Obviously, $\kappa(a \wedge b) \leq \kappa(a) \wedge \kappa(b)$. Suppose that $s \in \kappa(a) \wedge \kappa(b) = \kappa(a) \cap \kappa(b)$. This gives $a, b \in \xi(s)$, hence $a_s \leq a$, $a_s \leq b$, consequently $a_s \leq a \wedge b$, i.e. $s \in \kappa(a \wedge b)$.

(ii) Assume that $p \in \kappa(a)'$, where $a \in \mathcal{L}_0$ with $0 < a < 1$. Then $p \perp q$ for all $q \in \kappa(a)$. It follows that $\forall q \in \kappa(a), a_q \leq a'_p$. Hence $a_p \leq (\bigvee_{q \in \kappa(a)} a_q)' = a'$. This proves $\kappa(a)' \leq \kappa(a')$.

Now let $p \in \kappa(a')$, then $a_p \leq a' = (\bigvee \{a_q : a_q \leq a\})'$, hence $a_p \leq a'_q$ for all $q \in \kappa(a)$, which entails $p \in \kappa(a)'$.

The rest follows by Proposition 7. \square

4. SUPERPOSITION PRINCIPLE AND SECTORS

Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that property (A) and 3-MSP are satisfied. By Theorem 1, $(\Sigma, *)$, where $p * s = \lambda\{p, s\} = \{p, s\}^-$ is a projective geometry.

Definition 9. We will say that a superposition principle (SP, for short) is satisfied in $(\mathcal{L}, \Sigma, \xi)$, if for every $p, q \in \Sigma$, $p \neq q$, there is $r \in \{p, q\}^-$ such that $r \neq p, r \neq q$.

The following statement is straightforward.

Theorem 10. Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that (A), 3-MSP and SP are satisfied. The $(\Sigma, *)$ is an irreducible projective geometry.

The notion of a sector was introduced in [68] (see also [55, Definition 3.2.7]). Roughly speaking, a sector is a maximal λ -closed subset of Σ in which SP holds.

Definition 10. A nonempty subset $S \subset \Sigma$ is called a sector if the following conditions hold:

- (i) $S \in \mathcal{L}(\Sigma)$;
- (ii) for any two different $p, q \in S$ we can find $r \in \{p, q\}^-$ distinct from p and q ;
- (iii) if $q \in \Sigma \setminus S$, then $\{p, q\}^- = \{p, q\}$ for every $p \in S$.

A basic property of sectors is the following.

Lemma 7. If S, P are sectors, then either $S = P$ or $S \cap P = \emptyset$.

Proof. Assume that $S \neq P$. Then there is $q \in S \setminus P$ (or $q \in P \setminus S$), and by (ii) of Definition 10, $\{s, q\}^- \neq \{s, q\}$ whenever $s \in S \cap P$, while by (iii) of Definition 10, $\{s, q\}^- = \{s, q\}$. This contradiction implies that $S \cap P = \emptyset$. \square

Theorem 11. [55] Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that (A) and 3-MSP are satisfied. Then Σ can be written as a set theoretical union of sectors.

Proof. Let us define a binary relation \approx on Σ as follows: (i) for every $s \in \Sigma$, $s \approx s$, (ii) for distinct $s, t \in \Sigma$, $s \approx t$ if there is $r \in \{s, t\}^-$, $r \neq s, r \neq t$. We will prove that \approx is an equivalence relation. Reflexivity and symmetry are clear from the definition. To prove transitivity, assume that $p \approx r$ and $r \approx s$. With no loss of generality, we may assume that p, r, s are mutually different. Let $x \in \{p, r\}^- \setminus \{p, r\}$, $y \in \{r, s\}^- \setminus \{r, s\}$. By 2-MSP we have $\{p, r\}^- = \{p, x\}^- = \{r, x\}^-$, $\{r, s\}^- = \{r, y\}^- = \{s, y\}^-$. Moreover, $r \in \{x, p\}^-$ implies $y \in \{x, p, s\}^- = \lambda\{x, p, s\} \subset S$ by 3-MSP. If $y \in \{p, s\}^-$ and $y \neq p$, then y is a minimal superposition of $\{p, s\}$, and hence $p \approx s$. If $y = p$, then $p \in \{r, s\}^-$ implies $r \in \{s, p\}^-$, hence $p \approx s$. If $y = x$, then $\{r, x\}^- = \{r, y\}^-$ implies $\{p, r\}^- = \{r, s\}^-$, $p \in \{r, s\}^-$, hence $r \in \{s, p\}^-$ and $p \approx s$. Finally, if y is a minimal superposition, then $\{y, x\}^- \cap \{p, s\}^- \neq \emptyset$ implies that $p \approx s$.

Let \hat{s} denote the equivalence class containing $s \in \Sigma$. We may write $\Sigma = \bigcup \{\hat{s} : s \in \Sigma\}$. It can be easily seen that \hat{s} is a sector for every $s \in \Sigma$. \square

Sectors can be characterized by the closure operator λ as follows.

Theorem 12. *Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that (A) and 3-MSP are satisfied. Let $\Sigma = \bigcup_i S_i$, where $S_i \in \mathcal{L}(\Sigma)$ and (SP) is satisfied on S_i , $\forall i$. Then S_i are sectors if and only if they are λ -clopen sets.*

Proof. By Theorem 11, Σ can be covered by sectors, which are λ -closed. Let S be a sector. To prove that S is clopen, it suffices to prove that $\Sigma \setminus S$ is λ -closed. Assume that $p, q \in \Sigma \setminus S$ and let r be a minimal superposition of p, q . If $r \in S$, then by 2-MSP, $p \in \lambda\{r, q\}$. Since $r \in S$ and $q \notin S$, and S is a sector, we have $\lambda\{r, q\} = \{r, q\}$, which is a contradiction. Therefore $r \in \Sigma \setminus S$. This proves that sectors are λ -clopen sets.

Conversely, let S be a λ -clopen set such that SP is satisfied on S . Then conditions (i) and (ii) of Definition 10 are satisfied. To prove (iii), assume that $p \in S, q \notin S$, and $r \in \{p, q\}^-$, $r \neq p, q$, then either $r \in S$ or $r \notin S$. If $r \in S$, we get $q \in \{r, p\}^-$, which contradicts the supposition that S is λ -closed. If $r \notin S$, we get $p \in \{r, q\}^-$, which contradicts the supposition that S is open. It follows that $\{p, q\}^- = \{p, q\}$, hence S is a sector. \square

Definition 11. *We say that an element $a \in \mathcal{L}$ is classical (or a superselection rule) if there is an element $a' \in \mathcal{L}$ such that for every $s \in \Sigma$, $a \in \xi(s) \Leftrightarrow a' \notin \xi(s)$.*

Clearly, 0 and 1 are classical elements.

Theorem 13. *Let $a \in \mathcal{L}$ be a classical element. Then $\kappa(a) = \{s \in \Sigma : a \in \xi(s)\}$ is a clopen set in $\mathcal{F}(\Sigma)$.*

Proof. We have $\Sigma = \{s : a \in \xi(s)\} \cup \{s : a' \in \xi(s)\}$. By symmetry, it suffices to prove that $S := \{s : a \in \xi(s)\}$ belongs to $\mathcal{F}(\Sigma)$. It easily follows from the fact that $r \in S$ iff $\bigcap \{\xi(s) : s \in S\} \subset \xi(r)$, which entails that if $a \in \xi(s) \forall s \in S$, then $a \in \xi(r)$, hence $r \in S$. \square

From the fact that $\mathcal{F}(\Sigma) \subset \mathcal{L}(\Sigma)$, we obtain that $\kappa(a)$ is clopen also in $\mathcal{L}(\Sigma)$.

We recall that an element z in a lattice L with 0 and 1 is *central* when there exist two lattices L_1 and L_2 and an isomorphism between L and the direct product $L_1 \times L_2$ such that z corresponds to the element $(1_1, 0_2) \in L_1 \times L_2$. (cf e.g. [67, Definition (4.12)]. Evidently 0 and 1 are central elements.

Lemma 8. [67, Theorem(4.13)] *An element z of a lattice L with 0 and 1 is central if and only if there is an element z' in L such that*

$$(21) \quad a = (a \wedge z) \vee (a \wedge z') = (a \vee z) \wedge (a \vee z') \text{ for every } a \in L.$$

If L is orthocomplemented, then z is central if and only if the first equality in (21) is satisfied for every $a \in L$ ([67, Lemma (29.9)]).

Theorem 14. *Let $(\mathcal{L}, \mathcal{L}_0, \Sigma, \xi)$ be a state-property system such that conditions (A), (B), (C) are satisfied. Then an element $c \in \mathcal{L}$ is central if and only if c is classical.*

Proof. If properties (A), (B), (C) are satisfied, then \mathcal{L} is a complete, atomistic, orthocomplemented lattice, and $\kappa : \mathcal{L} \rightarrow \mathcal{F}(\Sigma)$ is an isomorphism (Theorem 9).

Let c be a central element of L , then by (21), for every atom $a \in \mathcal{L}$, $a = (a \wedge c) \vee (a \wedge c')$, hence either $a = a \wedge c$, or $a = a \wedge c'$. By Proposition 7, $\kappa(a) = \{s\}$

for some $s \in \Sigma$. Moreover, $\kappa(a) = \kappa(a \wedge c) \vee \kappa(a \wedge c')$, hence either $\kappa(a \wedge c) = \{s\}$, or $\kappa(a \wedge c') = \{s\}$, that is, either $c \in \xi(s)$ or $c' \in \xi(s)$. This entails that c is classical.

Conversely, if c is classical, i.e., for every $s \in \Sigma$, either $s \in \kappa(c)$ or $s \in \kappa(c')$, then for every $a \in \mathcal{L}$,

$$\begin{aligned} a &= \bigvee \{a_s : s \in \kappa(a)\} \\ &= \bigvee \{a_s : s \in \kappa(a) \cap \kappa(c)\} \vee \bigvee \{a_s : s \in \kappa(a) \cap \kappa(c')\} \\ &= \bigvee \{a_s : s \in \kappa(a \wedge c)\} \vee \bigvee \{a_s : s \in \kappa(a \wedge c')\}, \end{aligned}$$

and consequently, $a = (a \wedge c) \vee (a \wedge c')$. By Lemma 8, c is central element of \mathcal{L} . \square

5. CLOSED SUBSPACES AND MACKEY PROPERTY

Throughout this section we will use the following notations:

$$(22) \quad \text{For any } A, B \in \mathcal{L}(\Sigma), \quad A \sqcup B := \lambda(A \cup B).$$

$$(23) \quad \text{For any } A, B \in \mathcal{F}(\Sigma), \quad A \vee B := (A \cup B)^-.$$

For infima in both $\mathcal{L}(\Sigma)$, $\mathcal{F}(\Sigma)$ we use the same notation $A \wedge B (= A \cap B)$.

In [56], the following definitions were introduced, and the equivalence of the following three categories was proved.

Definition 12. A Mackey geometry is a projective geometry G together with a subset \mathcal{S} of subspaces of G satisfying the following axioms:

- (i) $\mathcal{A} \subseteq \mathcal{S}$ implies $\bigcap \mathcal{A} \in \mathcal{S}$ (hence \mathcal{S} is an intersection system),
- (ii) $\emptyset \in \mathcal{S}$,
- (iii) if $E \in \mathcal{S}$, then $a \vee E \in \mathcal{S}$ for every $a \in G$.

The elements of \mathcal{S} are called the closed subspaces of G . An isomorphism of Mackey geometries is an isomorphism of projective geometries $g : G_1 \rightarrow G_2$ satisfying $S \in \mathcal{S}_1$ iff $g(E) \in \mathcal{S}_2$ (where E is any subspace of G_1).

Definition 13. A Mackey lattice is a projective lattice L together with an operator $x \rightarrow c(x)$ satisfying the following axioms:

- (i) $x \leq c(x)$ for every $x \in L$,
- (ii) $x \leq c(y)$ implies $c(x) \leq c(y)$,
- (iii) $c(0) = 0$,
- (iv) if $x = c(x)$, then $a \vee x = c(a \vee x)$ for every atom a in L .

An element $x \in L$ is closed if $x = c(x)$. An isomorphism of Mackey lattices is an isomorphism of (projective) lattices $h : L_1 \rightarrow L_2$ satisfying $h(c_1(x)) = c_2(h(x))$ for every element $x \in L_1$.

For any lattice L we shall denote by A_L the set of all atoms of L . We say that a lattice L has the *intersection property* (cf [56, Definition 2.5.1]) if one has

$$(24) \quad a, b \in A_L, a \neq b, x \in L \text{ and } a \leq b \vee x \Rightarrow \exists c \in A_L \text{ with } c \leq (a \vee b) \wedge x.$$

If L is an atomistic lattice, the following conditions are equivalent:

- (1) L is upper and lower semimodular¹,

¹A lattice L is called *upper semimodular* if $u \wedge v \leq v$ implies $u \leq u \vee v$, and L is *lower semimodular* if $u \leq u \vee v$ implies $u \wedge v \leq v$. Here $a \leq b$ means that b covers a .

- (2) L has the covering property²
- (3) L has the intersection property.

Moreover, the implications $1 \Rightarrow 2 \Rightarrow 3$ hold for any lattice.

Definition 14. *An intersection lattice is a complete atomistic lattice C having the intersection property. (Equivalently, C is both upper and lower semimodular.)*

Let L_1 and L_2 be Mackey lattices. We say that a morphism $f : L_1 \rightarrow L_2$ is *continuous* if

$$(25) \quad f(c_1(x)) \leq c_2(f(x)) \text{ for every } x \in L_1.$$

Theorem 15. [56, Theorem 13.3.8] *The categories of Mackey geometries, of Mackey lattices and of intersection lattices are equivalent. This means that one has a functor \mathcal{L} from Mackey geometries to Mackey lattices, a functor \mathcal{C} from Mackey lattices to intersection lattices, a functor \mathcal{G} from intersection lattices to Mackey geometries, and natural isomorphisms $G \cong \mathcal{G}(\mathcal{C}(\mathcal{L}(G)))$, $L \cong \mathcal{L}(\mathcal{G}(\mathcal{C}(L)))$ and $C \cong \mathcal{C}(\mathcal{L}(\mathcal{G}(C)))$.*

In our setting, we obtain the following result.

Theorem 16. *Let $(\mathcal{L}, \mathcal{L}_0, \Sigma, \xi)$ satisfy properties (A), (B), (C) and 3-MSP. Then $\mathcal{L}(\Sigma)$ with the closure operation $c(A) = \bar{A}$, $A \in \mathcal{L}(\Sigma)$, is a Mackey lattice.*

Proof. (cf [55, Proposition 3.3.18]). Properties (i) -(iii) of Definition 13 are clear. We have to prove only property (iv).

By Theorem 8, $\mathcal{F}(\Sigma)$ is a complete, atomistic, orthocomplemented lattice with the orthocomplementation $S \mapsto S'$. We will use the fact that $\mathcal{L}(\Sigma) \supset \mathcal{F}(\Sigma)$, and $\mathcal{L}(\Sigma)$ is modular (Theorem 2). We must show that if $S \in \mathcal{F}(\Sigma)$ and $p \in \Sigma \setminus S$, $p \vee S = p \sqcup S$. Modularity of $\mathcal{L}(\Sigma)$ implies that $S \prec S \sqcup p$ (that is, $S \sqcup p$ covers S). Dually, $(S \sqcup p)' = S' \wedge \{p\}' \prec S'$. Then there is an atom $q \in \Sigma$ such that $(S \sqcup \{p\})' \sqcup \{q\} = S'$. Then $((S \sqcup \{p\})' \sqcup \{q\})' = S'' = S \prec (S \sqcup \{p\})''$. This entails $(S \sqcup \{p\})'' = S \vee \{p\} = S \sqcup \{p\}$. \square

Corollary 2. *If $(\mathcal{L}, \mathcal{L}_0, \Sigma, \xi)$ satisfy properties (A), (B), (C) then for every $S \in \mathcal{F}(\Sigma)$ and a finite dimensional element $P = \{p_1, \dots, p_n\}^-$, we have $S \vee P = S \sqcup P$.*

In accordance with theorems 15 and 16, if a state property system $(\mathcal{L}, \Sigma, \xi)$ satisfies conditions (A), (B), (C) and 3-MSP, we may consider Σ with elements of \mathcal{F} as closed subspaces as Mackey geometry, $\mathcal{L}(\Sigma)$ with the operator $S \mapsto \bar{S}$ as a Mackey lattice, and $\mathcal{F}(\Sigma)$ as intersection lattice. Indeed, by theorem 16, $\mathcal{L}(\Sigma)$ with the operation $S \mapsto \bar{S}$ is a Mackey lattice. By [56, Proposition 13.2.7], the set $\mathcal{F}(\Sigma)$ is an intersection lattice for the induced order. The infimum of any subset $\mathcal{A} \subset \mathcal{F}(\Sigma)$ is the element $\bigwedge \mathcal{A}$ and the supremum is $\bigvee \mathcal{A} = (\bigcap \mathcal{A})^-$. Moreover, the atoms of $\mathcal{F}(\Sigma)$ are the atoms of $\mathcal{L}(\Sigma)$, that is, elements of Σ . Further, $\mathcal{F}(\Sigma)$ being an intersection lattice, the set of all atoms Σ of $\mathcal{F}(\Sigma)$ is a projective geometry (cf [56, 2.5.7] and Theorem 1), and the set of closure subspaces coincides with the sets $\{F \subset \Sigma : F \in \mathcal{F}(\Sigma)\}$ as closed subspaces. Owing the isomorphism between $\mathcal{F}(\Sigma)$ and \mathcal{L} , the lattice \mathcal{L} can be considered as an intersection lattice with the atoms $\{a_s \in L : s \in \Sigma\}$.

In the sequel, we will need the following definition.

²A lattice L has the *covering property* if for $x \in L$ and any atom $a \in L$ one has, $a \wedge x = 0 \implies x \leq a \vee x$.

Definition 15. [56, Definition 13.4.6] A Mackey lattice L is called *regular* if for every closed element $x \in L$ and every atom $a \not\leq x$, there exists a closed coatom $h \in L$ such that $x \leq h$ and $a \not\leq h$.

6. ORTHOGEOMETRIES, ORTHOLATTICES AND ORTHOSYSTEMS

Definition 16. [56, Definition 14.1.1] An orthogeometry is a projective geometry with a relation \perp , called orthogonality, which satisfies the following axioms:

- (1) (O_1) $a \perp b$ implies $b \perp a$,
- (2) (O_2) if $a \perp p$, $b \perp p$ and $c \in a * b$, then $c \perp p$,
- (3) (O_3) if $a, b, c \in G$ and $b \neq c$, then there is $p \in b * c$ with $p \perp a$,
- (4) (O_4) for every $a \in G$ there exists $b \in G$ with $a \not\perp b$.

An isomorphism of orthogeometries is an isomorphism of projective geometries $g : G_1 \rightarrow G_2$ satisfying $a \perp b$ iff $ga \perp gb$.

For any subset $A \subseteq G$ the *orthogonal set* $A^\perp := \{x \in G : x \perp a \text{ for every } a \in A\}$ is a subspace of G by condition (O_2). A point a of an orthogeometry G is called a *null point* if $a \in a^\perp$. The geometry is called *non-null* if it contains a non-null point and *pure* if every point is non-null [56, Definition 14.1.7].

Definition 17. An ortholattice³ is a projective lattice together with an operator $x \mapsto x^\perp$ which satisfies the following conditions:

- (1) $x \leq x^{\perp\perp}$ for every $x \in L$,
- (2) $x \leq y$ implies $y^\perp \leq x^\perp$,
- (3) $0^{\perp\perp} = 0$,
- (4) if $x = x^{\perp\perp}$, then $a \vee x = (a \vee x)^{\perp\perp}$ for every atom $a \in L$.

An isomorphism of ortholattices is an isomorphism of lattices $f : L_1 \rightarrow L_2$ such that $f(x^\perp) = (fx)^\perp$ for every element $x \in L_1$.

Proposition 8. If L is an ortholattice, then L together with the operator $c(x) := x^{\perp\perp}$ is a regular Mackey lattice.

Definition 18. An orthosystem is an intersection lattice C together with an operator $x \mapsto x'$ satisfying the following conditions:

- (1) $x = x''$ for every $x \in C$,
- (2) $x \leq y$ implies $y' \leq x'$.

An isomorphism of orthosystems is an isomorphism of lattices $h : C_1 \rightarrow C_2$ such that $h(x') = (hx)'$ for every $x \in C_1$.

By [56, Remark 14.2.7], instead of an intersection lattice it is enough to require that C is a complete atomistic lattice satisfying the exchange property.

In [56], it is proved that there is a triple correspondence between orthogeometries and ortholattices and orthosystems [56, Proposition 14.2.11]. We summarize the results in the next theorem.

Theorem 17. (1) If L is an orthogeometry, then the projective lattice $\mathcal{L}(G)$ together with the operator $E \mapsto E^\perp$ is an ortholattice [56, Proposition 14.2.5].

³Please do not mistake it with orthocomplemented lattice, which is sometimes also called ortholattice.

- (2) Let L be an ortholattice. Denote by $\mathcal{C}(L)$ the set of all closed element $x = x^{\perp\perp}$ of L . Then $\mathcal{C}(L)$ together with the operator $x \mapsto x^\perp$ is an orthosystem (for the induced order) [56, Proposition 14.2.8].
- (3) Let C be an orthosystem. Then the projective geometry $\mathcal{G}(C)$ consisting of the set A_C of all atoms of C and the operator $*$, $a * b = \{c \in A_C : c \leq a \vee b\}$, together with the relation $a \perp b$ iff $a \leq b'$, is an orthogeometry [56, proposition 14.2.9].

Theorem 18. *Let $(\mathcal{L}, \mathcal{L}_0, \Sigma, \xi)$ satisfy properties (A), (B), (C) and 3-MSP. Then $\mathcal{F}(\Sigma)$ is an orthosystem.*

Proof. By Theorems 16 and 15, $\mathcal{F}(\Sigma)$ is an intersection lattice. By Theorem 8, the mapping $S \mapsto S'$ is an orthocomplementation on $\mathcal{F}(\Sigma)$, which implies the desired result. \square

Remark 1. *If $(\mathcal{L}, \mathcal{L}_0, \Sigma, \xi)$ satisfies properties (A), (B), (C) and 3-MSP then, according to Theorem 17, (Σ, λ) is an orthogeometry and $\mathcal{L}(\Sigma)$ with the closed subspaces $\mathcal{F}(\Sigma)$ is an ortholattice. Moreover, since $\bar{S} = S''$, in accordance with [56, Proposition 14.2.4], $\mathcal{F}(\Sigma)$ is a regular Mackey lattice.*

7. REPRESENTATIONS IN VECTOR SPACES

Let V be any vector space over a field K . We emphasize that the dimension of V is arbitrary (possibly infinite) and K is allowed to be a skew field (often called division ring).

Proposition 9. [56, Proposition 2.1.6]. *Let V be any vector space. On $V^* := V \setminus \{0\}$ one defines a binary relation as follows: $x \sim y$ iff x, y if x, y are linearly dependent. Since this is an equivalence relation, the quotient set $\mathcal{P}(V) := V^* / \sim$ is well defined and becomes a projective geometry if for any elements $X, Y, Z \in \mathcal{P}(V)$ one defines $\ell(X, Y, Z)$ iff X, Y, Z have linearly dependent representatives x, y, z .*

Theorem 19. *Let G be an irreducible projective geometry containing at least four independent points. Then there exists a (left) vector space V over a field K such that G is isomorphic to $\mathcal{P}(V)$.*

Definition 19. [56, Definition 14.1.5] *Let V be a vector space over K . A map $\Phi : V \times V \rightarrow K$ is called a reflexive (or also symmetric) sesquilinear form if there exists an anti-isomorphism of fields $\sigma : K \rightarrow K$ such that the following axioms are satisfied:*

- (1) $\Phi(x_1 + x_2, y) = \Phi(x_1, y) + \Phi(x_2, y)$ and $\Phi(\lambda x, y) = \lambda \cdot \Phi(x, y)$,
- (2) $\Phi(x, y_1 + y_2) = \Phi(x, y_1) + \Phi(x, y_2)$ and $\Phi(x, \mu y) = \Phi(x, y) \cdot \sigma(\mu)$;
- (4) $\Phi(x, y) = 0$ iff $\Phi(y, x) = 0$.

A map $\Phi : V \times V \rightarrow K$ is called a Hermitian form if there exists an involution $\sigma : K \rightarrow K$, i.e., an anti-isomorphism of order 2, such that the following axioms are satisfied:

- (1) $\Phi(x - 1 + x_2, y) = \Phi(x_1, y) + \phi(x_2, y)$ and $\Phi(\lambda x, y) = \lambda \cdot \Phi(x, y)$,
- (4) $\Phi(x, y) = \sigma(\Phi(y, x))$ for all $x, y \in V$.

Obviously, these two axioms imply both (2) and (3). Finally, we recall that the form Φ is non-singular if $\Phi(x, y) = 0$ for all $y \in V$ implies $x = 0$.

Proposition 10. [56, Proposition 14.1.6] *If $\Phi : V \times V \rightarrow K$ is a non-singular reflexive sesquilinear form, then the projective geometry $\mathcal{P}(V)$ together with the relation \perp defined by $[x] \perp [y]$ iff $\Phi(x, y) = 0$ is an orthogeometry.*

Definition 20. *A point a of an orthogeometry G is called a null point if $a \in a^\perp$. The orthogeometry is called non-null if it contains a non-null point and pure if every point is non-null.*

Let V be a pre-Hilbertian space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, then trivially the associated orthogeometry $\mathcal{P}(V)$ is pure.

Theorem 20. [56, Theorem 14.1.8] *Let V be a vector space of dimension ≥ 3 over a field K , and suppose that $\mathcal{P}(V)$ together with the relation \perp is an orthogeometry. Then there exists a non-singular reflexive sesquilinear form $\Phi : V \times V \rightarrow K$ which induces the orthogonality \perp in the sense of Proposition 10. Moreover, if $\mathcal{P}(V)$ is non-null, then \perp can be induced by a (non-singular) Hermitian form.*

We call states $S := \{s_1, \dots, s_n\}$ in Σ independent if $\forall i, s_i \notin \lambda(S \setminus s_i)$.

Theorem 21. *Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that conditions (A), SP, 3-MSP are satisfied. Assume that there exist at least four independent states in Σ . Then there is a field K and a vector space V over K such that the set $\mathcal{L}(\Sigma)$ of all linear subspaces of Σ is isomorphic to the lattice $\mathcal{L}(V)$ of all linear subspaces of V .*

Theorem 22. *Let $(\mathcal{L}, \Sigma, \xi)$ be a state property system such that conditions (A), (B), (C), SP, 3-MSP are satisfied. Assume that there exist at least four independent states in Σ . Then there exists a field K , an involutive ant-automorphism $*$: $K \rightarrow K$, a vector space V over K and a Hermitian form $f : V \times V \rightarrow K$ such that $\mathcal{F}(\Sigma)$ is orthoisomorphic to the set $\mathcal{L}_f(V)$ of all closed subspaces of V .*

(See [67] for the ideas of proof).

REFERENCES

- [1] Segal, I., Postulates for general quantum mechanics, *Annals of Mathematics* **48**, (1947), 930-948.
- [2] Emch, G. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley and Sons, New York, London, Toronto, Sydney, 1972.
- [3] Haag, R., Kastler, D., An algebraic approach to quantum field theory, *Journal of Mathematical Physics* **5** (1964), 841-861.
- [4] Alfsen, E., Schultz, F., On the geometry of noncommutative spectral theory, *Bull. Amer. Math. Soc.* **81** (1975), 893-895.
- [5] Edwards, C., The operational approach to algebraic quantum theory, *Communications in Mathematical Physics* **16** (1970), 207-230.
- [6] Edwards, C., Classes of operations in quantum theory, *Communications in Mathematical Physics* **20** (1971), 26-56.
- [7] Mielnik, B., Geometry of quantum states, *Communications in Mathematical Physics* **9** (1968), 55-80.
- [8] Mielnik, B., Theory of filters *Communications in Mathematical Physics* **15** (1969), 1-46.
- [9] Foulis, D. and Randall, C., What are quantum logics and what ought they to be? in *Current Issues in Quantum Logic*, eds. E. Beltrametti and B. van Fraassen, Kluwer Academic, Dordrecht, 35, 1981.
- [10] Randall, C. and Foulis, D., A mathematical setting for inductive reasoning, in *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science III*, C. Hooker, ed., Kluwer Academic, Dordrecht, 169, 1976.

- [11] Randall, C. and Foulis, D., The operational approach to quantum mechanics, in *Physical Theories as Logico-Operational structures*, Hooker, C.A. (ed.), Kluwer Academic, Dordrecht, 167, 1978.
- [12] Randall, C. and Foulis, D., Operational statistics and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., B.I. Wissenschaftsverlag, Bibliographisches Institut, Mannheim, 21, 1981.
- [13] Randall, C. and Foulis, D., Properties and operational propositions in quantum mechanics, *Found. Phys.*, **13**, 835, 1983.
- [14] Birkhoff, G. and Von Neumann, J., The logic of quantum mechanics, *Annals of Mathematics*, **37**, 823, 1936.
- [15] Zierler, N., Axioms for non-relativistic quantum mechanics, *Pacific Journal of Mathematics*, **11**, 1151, 1961.
- [16] Mackey, G.W., *Mathematical Foundations of Quantum Mechanics*, Benjamin, Reading, Mass., 1963.
- [17] Piron, C., Axiomatique quantique, *Helv. Phys. Acta*, **37**, 439, 1964.
- [18] Jauch, J., *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1968.
- [19] Varadarajan, V.S., *Geometry of Quantum Theory*, von Nostrand, Princeton, New Jersey, 1968.
- [20] Beltrametti, E. and Cassinelli, G., *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1981.
- [21] Wilbur, W., On characterizing the standard quantum logics, *Trans. Amer. Math. Soc.*, **233**, 265–282, 1977.
- [22] Keller, H., Ein nicht-klassischer Hilbertscher Raum, *Math. Z.*, **172**, 41–49, 1980.
- [23] Solèr, M. P., Characterization of Hilbert spaces by orthomodular spaces, *Comm. Algebra*, **23**, 219–243, 1995.
- [24] Holland, S.S. Jr., Orthomodularity in infinite dimensions; a theorem of M. Solèr, *Bull. Amer. Math. Soc.*, **32**, 205–234, 1995.
- [25] Pulmannova, S., Quantum-logics and Hilbert-space, *Found. Phys.*, **24**, 1403–1414, 1994.
- [26] Pulmannova, S., Axiomatization of quantum logics, *Int. J. Theor. Phys.*, **35**, 2309–2319, 1996.
- [27] Aerts, D., Van Steirteghem, B., Quantum axiomatics and a theorem of M.P. Soler, *Int. J. Theor. Phys.*, **39**, 497–502, 2000.
- [28] Ischi, B., Endomorphisms of the separated product of lattices, *International Journal of Theoretical Physics*, **39**, 2559–2581, 2000.
- [29] Ischi, B., Generalized quantum mechanics for separated systems, *Foundations of Physics Letters*, **14**, 501–518, 2001.
- [30] Ischi, B., Property lattices for independent quantum systems, *Reports on Mathematical Physics*, **50**, 155–165, 2002.
- [31] Ischi, B., A characterization of the Aerts product of Hilbertian lattices, *Reports on Mathematical Physics*, **56**, 39–51, 2005.
- [32] Watanabe, T., A connection between distributivity and locality in compound P-lattices, *Journal of Mathematical Physics*, **44**, 564–569, 2003.
- [33] Watanabe, T., Locality and orthomodular structure of compound systems, *Journal of Mathematical Physics*, **45**, 1795–1803, 2004.
- [34] Niestegge, G., Composite systems and the role of the complex numbers in quantum mechanics, *Journal of Mathematical Physics*, **45**, 4714–4725, 2004.
- [35] Piron, C., *Foundations of Quantum Physics*, Reading, Mass., W. A. Benjamin, 1976.
- [36] Aerts, D., *The one and the many*, Doctoral Thesis, Free University of Brussels, Brussels, 1981.
- [37] Aerts, D., Description of many physical entities without the paradoxes encountered in quantum mechanics, *Found. Phys.*, **12**, 1131, 1982.
- [38] Aerts, D., Classical theories and non classical theories as a special case of a more general theory, *J. Math. Phys.*, **24**, 2441, 1983.
- [39] Aerts, D., The description of one and many physical systems, in *Foundations of Quantum Mechanics*, eds. C. Gruber, A.V.C.P., Lausanne, 63, 1983.

- [40] Piron, C., Recent developments in quantum mechanics, *Helv. Phys. Acta*, **62**, 82, 1989.
- [41] Piron, C., *Mécanique Quantique: bases et applications*, Press Polytechnique de Lausanne, 1990.
- [42] Ludwig, G., *Foundation of Quantum Mechanics I*, Springer Verlag, Berlin, 1983.
- [43] Ludwig, G., *Foundation of Quantum Mechanics II*, Springer Verlag, Berlin, 1985.
- [44] D. Aerts, Foundations of quantum physics: a general realistic and operational approach, *Int. J. Theor. Phys.*, **38**, 289–358, 1999.
- [45] D. Aerts, Quantum mechanics: structures, axioms and paradoxes, in *Quantum Mechanics and the Nature of Reality*, eds. Aerts, D. and Pykacz, J., Kluwer Academic, Dordrecht, 1999.
- [46] Aerts, D., Colebunders, E., Van der Voorde, A., Steirteghem, B. State property systems and closure spaces: a study of categorical equivalence, *Inter. J. Theor. Phys.* **38**, 359–385, 1999.
- [47] D. Aerts, E. Colebunders, A. Van der Voorde and B. Van Steirteghem, The construct of closure spaces as the amnesic modification of the physical theory of state property systems, *Applied Categorical Structures*, **10**, 469–480, 2002.
- [48] Van Steirteghem, B., *Quantum Axiomatics: Investigation of the Structure of the Category of Physical Entities and Solér's Theorem*, Dissertation for the degree of Bachelor in Science, Brussels Free University, 1998.
- [49] Van Steirteghem, B., T_0 separation in axiomatic quantum mechanics, *Int. J. Theor. Phys.*, **39**, 955, 2000.
- [50] Van der Voorde, A., A categorical approach to T_1 separation and the product of state property systems, *Int. J. Theor. Phys.*, **39**, 947–953, 2000.
- [51] Van der Voorde, A., *Separation Axioms in Extension Theory for Closure Spaces and Their Relevance to State Property Systems*, Doctoral Thesis, Brussels Free University, 2001.
- [52] Aerts, D., Van der Voorde, A. and Deses, D., Connectedness applied to closure spaces and state property systems, *Journal of Electrical Engineering*, **52**, 18–21, 2001.
- [53] D. Aerts and D. Deses, State property systems and closure spaces: extracting the classical and nonclassical parts, in *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, eds. D. Aerts, M. Czachor and T. Durt, World Scientific, Singapore, 2002.
- [54] Aerts, D., Van der Voorde, A. and Deses, D., Classicality and connectedness for state property systems and closure spaces, to be published in *International Journal of Theoretical Physics*.
- [55] Pták, P., Pulmannová, S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1981.
- [56] Faure, C-A., Frölicher, A., *Modern Projective Geometry*, Kluwer, Dordrecht, 2000.
- [57] Aerts, D. and Daubechies, I., Physical justification for using the tensor product to describe two quantum systems as one joint system. *Helvetica Physica Acta*, **51**, 661–675, 1978.
- [58] Aerts, D. and Daubechies, I. Characterization of subsystems in physics. *Letters in Mathematical Physics*, **3**, 11–17, 1979.
- [59] Aerts, D. and Daubechies, I. Mathematical condition for a sub-lattice of a propositional system to represent a physical subsystem with a physical interpretation. *Letters in Mathematical Physics*, **3**, 19–27, 1979.
- [60] Aerts, D., Subsystems in physics described by bi-linear maps between the corresponding vector spaces. *Journal of Mathematical Physics*, **21**, 778–788, 1980.
- [61] Pulmannová, S., Coupling of quantum logics, *International Journal of Theoretical Physics*, **22**, 837–850, 1983.
- [62] Aerts, D., Construction of a structure which makes it possible to describe the joint system of a classical and a quantum system. *Reports in Mathematical Physics*, **20**, 421–428, 1984.
- [63] Aerts, D., Construction of the tensor product for lattices of properties of physical entities. *Journal of Mathematical Physics*, **25**, 1434–1441, 1984.
- [64] Pulmannová, S., On the products of quantum logics, *Suppl. Rendiconti del Circolo Matematico di Palermo, Ser. II*, 231–235, 1984.

- [65] Pulmannová, S., Tensor product of quantum logics, *Journal of Mathematical Physics*, **26**, 1–5, 1985.
- [66] Varadarajan, V.S., *Geometry of Quantum Theory*, Springer, New York 1985
- [67] Maeda, F., Maeda, M., *Theory of Symmetric Lattices*, Springer-verlag, Berlin Heidelberg New York, 1970.
- [68] Pulmannová, S., A superposition principle in quantum logics, *Communications in Mathematical Physics*, **49**, 47–51, 1976.

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